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Capillary–gravity waves produced by a wavemaker

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Surface waves in a channel can be produced by the horizontal motion of a plane wavemaker at one end of the channel. The amplitude and the frequency of the waves depend on both surface tension and gravity, as well as on the condition imposed at the contact line between the free surface and the wavemaker. Some of the previous work on the generation of capillary–gravity waves has been based on the unjustified assumption that the slope of the free surface at the contact line can be prescribed. A more acceptable condition is one that relates the slope to the motion of the contact line relative to the wavemaker; in this way the dynamic properties of the contact angle can be incorporated. The waves generated by a plane wavemaker in fluid of infinite depth and in fluid of a depth equal to that of the wavemaker are determined. An important reason for including surface tension is that in its absence the transient motion initiated by an impulsive start is singular; when surface tension is included this singularity is removed.

1. Wave production by wavemakers

Waves on the free surface of a fluid in a gravitational field can be produced by the normal motion of a rigid plate immersed in the fluid. The displacement of the fluid by the plate leads to a deformation of the free surface, which propagates away from the plate. The wavemaker problem is that of determining the characteristics of this propagating wavetrain, given the motion of the wavemaker. If we suppose that the wavemaker oscillates with a given amplitude and frequency, the steady state at large distance from the plate will consist of a plane wave with the given frequency, and the amplitude and phase of this wave are the quantities to be determined; this calculation was first performed by Havelock (1929). If we suppose that the fluid is in a channel of finite depth and the wavemaker is at one end of the channel, several different motions of the vertical boundary can be considered. For example, the whole of the plane wall can be made to oscillate rigidly, either remaining vertical or being hinged at the bottom, or we could move only the top section of the boundary, the lower section being held at rest. The response of the fluid to motions with an arbitrary time-dependence, including the transient motion at the initiation of a harmonic oscillation, can be solved by means of a Laplace transformation. The solutions for a range of wavemaker velocities differing in their dependence on depth and time have been obtained by Faltas (1988). There is, however, a difficulty in determining the transient motion after an impulsive start, because an initial singularity in the slope of the free surface at the wavemaker is predicted. This phenomenon was described in an unpublished note by Peregrine in 1972, and is treated at length by Roberts (1987). He considered the transient motion for power-law motions of the wavemaker and concluded that the singularity could only be removed by starting the motion sufficiently smoothly.

The solutions so far described have ignored the presence of surface tension, which also acts to provide a restoring force on the free surface. The dispersion relation for waves controlled by the combination of gravity and surface tension is well known, and suffices to determines the properties of such capillary–gravity waves in the absence of vertical boundaries. Since, however, the presence of surface tension increases the order of the dynamic boundary condition on the pressure at the free surface, the problem of capillary–gravity waves in a horizontally bounded fluid is only closed when some statement has been made of the edge condition to be applied where the free surface and the boundary intersect. The need for this extra condition was first pointed out by Evans (1968) in his discussion of the reflection of capillary–gravity waves by a vertical barrier. The wavemaker problem with surface tension included was discussed by Rhodes-Robinson (1971) who assumed that the slope of the free surface at the edge could be prescribed and varied in phase with the horizontal motion of the wavemaker. The transient waves produced by the initial motion of the wavemaker are the subject of a recent paper by Joo, Schultz & Messiter (1990). These authors concentrate on the motion induced by a plane wavemaker of the same depth as the fluid, and on an impulsive acceleration (ramp) and impulsive velocity (step). They specifically include dynamic-contact-angle effects in their analysis, allowing the varying slope of the free surface at the contact line to be a known but unspecified function of the time. Their results, however, are all for a fixed contact angle. For the ramp motion of the wavemaker they encounter no singularity in the free-surface elevation, which they determine for small values of both time and distance from the wavemaker. For the step motion they find that an initial singularity is still present, even though surface tension has been included. They conclude that the correct formulation for small time and distance requires the full nonlinear free-surface conditions.

This work, in common with other attempts at describing the transient motion for capillary–gravity waves generated by a wavemaker, is unsatisfactory because it assumes that it is possible to prescribe what slope the free surface should have. In general, there is no mechanism by which the slope can be controlled. The exception is when the contact angle remains fixed, which is a dynamically possible situation, valid in the limit in which there is no dynamic variation of the contact angle, or when this variation is so small that it can safely be neglected. Since it is known that the contact angle at a moving contact line between a fluid and a solid varies with the speed of the contact line, we can prescribe this variation and so allow the free surface to respond to the motion of the contact line relative to the wavemaker. A condition of this kind has been used to determine the amplitudes of capillary–gravity waves generated by the vertical motion of a plate (Hocking 1987*a*) and the reflection of an incident wave by a fixed plate (Hocking 1987*b*) and by a circular cylinder (Mahdmina & Hocking 1990). We suppose, for simplicity, that the contact angle can vary about a value of 90°, the variation being proportional to the velocity of the contact line relative to the boundary. Extreme cases of this condition include the possibility of orthogonal contact, as is present in the absence of surface tension, and of a fixed contact line with a necessarily varying contact angle.

The wavemaker problems for capillary–gravity waves that are studied in this paper make use of this edge condition. We consider the particular case of a plane vertical wavemaker which is impulsively brought into a harmonic oscillation of small amplitude. We concentrate on two special cases: fluid of finite depth with the wavemaker extending from top to bottom of the fluid, and fluid of infinite depth with only the top portion of the vertical boundary of the fluid brought into motion. The amplitude of the steady-state wavetrain is obtained, generalizing the results of Havelock (1929). More importantly, examination of the small-time solution shows that, when the postulated edge condition is employed, there is no singularity in the free-surface elevation or the slope at the wavemaker, even when it is started impulsively. It is not necessary to include nonlinear terms in the free-surface condition to arrive at an acceptable solution.

2. Formulation and non-dimensionalization

Consider inviscid fluid in a channel of depth d', with a wavemaker of immersed depth h' at one end of the channel. The wavemaker is made to oscillate with frequency σ' and with a small amplitude ϵ' about its mean position. The portion of the end of the channel below the wavemaker is fixed. Surface waves of frequency σ' in fluid of depth d' have a wavelength $2\pi/k'$, given by

$$\sigma'^2 = (gk'+\gamma k'^3)\tanh k'd', \tag{2.1}$$

where g is the gravitational acceleration and γ the surface tension. We choose $1/k'$ as the lengthscale for non-dimensionalization and $(gk')^{-\frac{1}{2}}$ as the timescale. Velocities are scaled with $\epsilon'(gk')^{\frac{1}{2}}$ and the pressure is written in the form

$$p' = p_0+\rho gz'+\epsilon'\rho gp, \tag{2.2}$$

where ρ is the density of the fluid and p_0 the pressure in the air above the channel. The horizontal x-coordinate is measured from the end of the channel and the vertical z-coordinate from the equilibrium free surface. The equations for small-amplitude waves with no variation across the width of the channel are satisfied by a potential ϕ that satisfies Laplace's equation, with the horizontal and vertical velocity components and the pressure being given by

$$u = \frac{\partial\phi}{\partial x}, \quad w = \frac{\partial\phi}{\partial z}, \quad p = -\frac{\partial\phi}{\partial t}. \tag{2.3}$$

The bottom condition on ϕ is

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = -d \ (d \text{ finite}), \quad \frac{\partial\phi}{\partial z}\rightarrow 0 \quad \text{as} \quad z\rightarrow -\infty \ (d \text{ infinite}), \tag{2.4}$$

where $d = k'd'$. The motion is forced by the wavemaker, and, if its position is taken as $\mathrm{i}\exp(-\mathrm{i}\sigma t)$, the conditions on ϕ at the end of the channel are, for $t > 0$,

$$\frac{\partial\phi}{\partial x} = \sigma\exp(-\mathrm{i}\sigma t) \quad \text{for} \quad 0 > z > -h, \quad \frac{\partial\phi}{\partial x} = 0 \quad \text{for} \ -h > z > -d, \tag{2.5}$$

where $h = k'h'$ and

$$\sigma^2 = (1+K)\tanh d. \tag{2.6}$$

Because the amplitude of the lateral displacement of the wavemaker is small, this condition can be applied at the mean position $x = 0$.

The elevation of the free surface is equal to $\epsilon'\eta(x,t)$ and the conditions at the free surface are

$$\frac{\partial\eta}{\partial t} = w, \quad \eta-K\frac{\partial^2\eta}{\partial x^2} = p; \tag{2.7}$$

since we are assuming waves of small amplitude these conditions can be applied at

$z = 0$. The parameter K measures the relative importance of capillarity and gravity and is defined by

$$K = \frac{\gamma k'^2}{\rho g}. \tag{2.8}$$

The edge condition to be applied at the intersection of the free surface and the wavemaker states that the slope of the free surface there is proportional to the speed of the contact line relative to the wavemaker. In non-dimensional form this condition becomes

$$\frac{\partial \eta}{\partial t} = \lambda \frac{\partial \eta}{\partial x} \quad \text{at} \quad x = 0. \tag{2.9}$$

If $\lambda = 0$ the contact line does not move relative to the wavemaker and if $\lambda = \infty$ the contact angle remains fixed at its static value of 90°. This completes the formulation of the problem to be solved.

The chosen form for the wavemaker velocity can be replaced by more general functions of time; it could also be allowed to vary with the depth below the free surface. A similar analysis to that presented here can be performed to deal with these variations. The oscillation of the wavemaker begins at $t = 0$ and the complex form of its velocity allows for both an initial impulsive velocity and an impulsive acceleration by taking the real or imaginary part of the solution, respectively. Because the analysis takes different forms for finite and infinite fluid depth, we treat the two cases separately.

3. Finite depth

For fluid of depth $d = h$, the whole end of the channel at $x = 0$ is made to move. We take a Laplace transform in t, with parameter s, and indicate the transform of ϕ by $\hat{\phi}$, for example. Then we can write $\hat{\phi} = \phi_1 + \phi_2$, where ϕ_2 satisfies the inhomogeneous condition (2.5) and the bottom condition (2.4). The value of ϕ_1 is given by

$$\phi_1 = \frac{-\sigma}{s + \mathrm{i}\sigma} \sum_0^\infty \frac{2(-1)^n}{hk_n^2} \cos\{k_n(z+h)\} \exp(-k_n x), \tag{3.1}$$

where

$$k_n = (n + \tfrac{1}{2})\,\pi/h. \tag{3.2}$$

The value of ϕ_2 that satisfies (2.4) and null conditions on $x = 0$ has the form

$$\phi_2 = \int_0^\infty \hat{A}(s,k) \frac{\cosh\{k(z+h)\}}{\cosh kh} \cos kx \,\mathrm{d}k. \tag{3.3}$$

The pressure associated with ϕ_1 is zero on the surface, and $\hat{\eta}$ can be found from the second part of (2.7) in the form

$$\hat{\eta} = -s \int_0^\infty \frac{\hat{A}}{1 + Kk^2} \cos kx \,\mathrm{d}k - K^{\frac{1}{2}} \hat{B}(s) \exp(-x/K^{\frac{1}{2}}), \tag{3.4}$$

where we have used the condition that $\hat{\eta}$ must be bounded at infinity. The slope of the free surface at the contact line is equal to $B(t)$. The second term can be written as a Fourier integral, so that

$$\hat{\eta} = -\int_0^\infty \frac{s\hat{A} + 2\hat{B}K/\pi}{1 + Kk^2} \cos kx \,\mathrm{d}k. \tag{3.5}$$

From the first condition in (2.7) we find that

$$s\hat{\eta} = \frac{\sigma}{s+\mathrm{i}\sigma}\sum_{0}^{\infty}\frac{2}{hk_n}\exp\left(-k_n x\right)+\int_0^\infty k\hat{A}\tanh kh\cos kx\,\mathrm{d}k, \tag{3.6}$$

and this too can be written as a single Fourier integral in the form

$$s\hat{\eta} = \int_0^\infty\left(k\hat{A}+\frac{\sigma}{s+\mathrm{i}\sigma}\frac{2}{\pi k}\right)\tanh kh\cos kx\,\mathrm{d}k. \tag{3.7}$$

When the two expressions (3.5) and (3.7) for $\hat{\eta}$ are equated, we find that $\hat{A}(s,k)$ can be written in terms of $\hat{B}(s)$ in the form

$$(s^2+\sigma_k^2)\hat{A} = -\frac{2Ks}{\pi}\hat{B}-\frac{2\sigma}{\pi(s+\mathrm{i}\sigma)}\frac{(1+Kk^2)\tanh kh}{k}, \tag{3.8}$$

where

$$\sigma_k^2 = k(1+Kk^2)\tanh kh, \quad \sigma_1 = \sigma. \tag{3.9}$$

The edge condition (2.9) provides another equation linking $\hat{A}$ and $\hat{B}$ in the form

$$\int_0^\infty\left(k\hat{A}+\frac{\sigma}{s+\mathrm{i}\sigma}\frac{2}{\pi k}\right)\tanh kh\,\mathrm{d}k = \lambda\hat{B}. \tag{3.10}$$

Hence we find that $\hat{B}$ can be determined from the equation

$$\left[\lambda+\frac{2Ks}{\pi}\int_0^\infty\frac{k\tanh kh}{s^2+\sigma_k^2}\mathrm{d}k\right]\hat{B} = \frac{2\sigma s^2}{\pi(s+\mathrm{i}\sigma)}\int_0^\infty\frac{\tanh kh}{k(s^2+\sigma_k^2)}\mathrm{d}k. \tag{3.11}$$

Inverting both sides of the equation, we find that $B(t)$ satisfies the following integral equation:

$$\lambda B(t)+\frac{2K}{\pi}\int_0^\infty k\tanh kh\int_0^t\cos\sigma_k\tau B(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}k$$
$$= \frac{2\sigma}{\pi}\int_0^\infty\frac{\tanh kh}{k}\left[\frac{\sigma_k^2\cos\sigma_k t-\sigma^2\cos\sigma t-\mathrm{i}\sigma(\sigma_k\sin\sigma_k t-\sigma\sin\sigma t)}{\sigma_k^2-\sigma^2}\right]\mathrm{d}k. \tag{3.12}$$

The slope of the free surface at the contact line is equal to $B(t)$ and the elevation of the free surface there can be found by inverting $\lambda\hat{B}/s$, so that

$$\eta(0,t) = \lambda\int_0^t B(\tau)\,\mathrm{d}\tau. \tag{3.13}$$

The transient motion introduced by the initial motion of the wavemaker is most easily found by considering (3.11) for large values of s. We find that, for finite λ, $\hat{B} = O(s^{-\frac{4}{3}}\ln s)$, so that $B(t) = O(t^{\frac{1}{3}}\ln t)$ as $t\to 0$. The fixed-contact-angle case is given formally by $\lambda = \infty$ and then $B(t) = 0$. The initial free-surface elevation can be found from (3.13) when λ is finite, and $\eta(0,t) = O(\lambda t^{\frac{4}{3}}\ln t)$. For $\lambda = \infty$ we can find $\lambda\hat{B}$ from (3.11) and then (3.13) shows that, in this case, $\eta(0,t) = O(t\ln t)$. It follows, therefore, that the edge condition used here does not introduce any initial singularity in either the free-surface elevation or slope at the contact line.

The solution for large t is dominated by the contribution from the pole at $s = -\mathrm{i}\sigma$. At this value of s the denominators of the two integrals in (3.11) become zero at $k = 1$, and the contour for both integrals with respect to k must be indented to lie below this singularity, since the contour in the s-plane must lie to the right of $-\mathrm{i}\sigma$. Hence

$$B(t) \sim \mathrm{e}^{-\mathrm{i}\sigma t}\frac{\sigma^2}{\mathrm{i}K}\left[\frac{J_1 - J_2 + \dfrac{\mathrm{i}\pi}{q}}{J_1 + \dfrac{\mathrm{i}\pi}{q} + \dfrac{\mathrm{i}\lambda\pi}{2\sigma K}}\right]. \tag{3.14}$$

The quantity q introduced in this expression is proportional to the group velocity of surface waves of frequency σ and is defined by

$$q = 1 + 3K + \frac{2h(1+K)}{\sinh 2h}. \tag{3.15}$$

The quantities J_1 and J_2 are integrals defined by

$$J_1 = \int_0^\infty \frac{k \tanh kh}{\sigma_k^2 - \sigma^2}\,\mathrm{d}k, \tag{3.16}$$

$$J_2 = \int_0^\infty \frac{(k^2-1)\tanh kh}{k(\sigma_k^2 - \sigma^2)}\,\mathrm{d}k. \tag{3.17}$$

Note that we take the principal value of the integral J_1 but that the integrand in J_2 is not singular.

The value of η for large t and large x can be found from (3.4) and the dominant contribution comes from the pole at $k = 1$ in the value of $\hat{A}$ given by (3.8), from which we see that, as $k \to 1$,

$$A(t,k) \sim \frac{2\sigma}{\pi}\left[\frac{\mathrm{i}KB(t) - \sigma^2\,\mathrm{e}^{-\mathrm{i}\sigma t}}{q \tanh h\,(k-1)}\right]. \tag{3.18}$$

In this way we find that, after the transients have disappeared, the wavemaker produces a wave whose elevation at large distance from the wavemaker has the form $R\exp\{\mathrm{i}(x-\sigma t)\}$, where

$$R = \frac{2\sigma^2}{q}\left[\frac{J_2 + \dfrac{\mathrm{i}\lambda\pi}{2\sigma K}}{J_1 + \dfrac{\mathrm{i}\pi}{q} + \dfrac{\mathrm{i}\lambda\pi}{2\sigma K}}\right]. \tag{3.19}$$

The free-surface elevation at the wavemaker in the steady state can be found from (3.11) and (3.13) and has the form

$$\eta(0,t) = \mathrm{e}^{-\mathrm{i}\sigma t}\frac{\lambda\sigma}{K}\left[\frac{J_1 - J_2 + \dfrac{\mathrm{i}\pi}{q}}{J_1 + \dfrac{\mathrm{i}\pi}{q} + \dfrac{\mathrm{i}\lambda\pi}{2\sigma K}}\right]. \tag{3.20}$$

The evaluation of $|R|$, the amplitude of the wave generated by the wavemaker, and

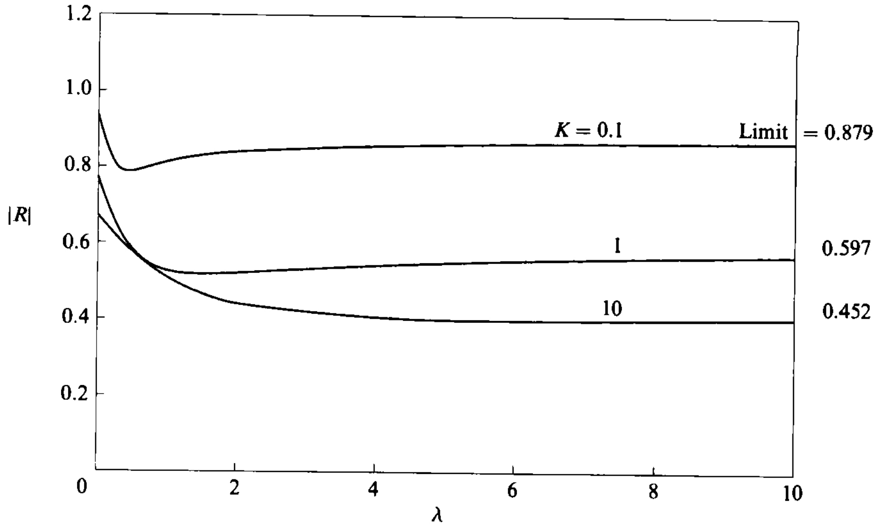

FIGURE 1. The amplitude of the wave generated by a wavemaker of depth h in fluid of depth h, when $h = 1$. The limiting values as $\lambda \to \infty$ are also shown.

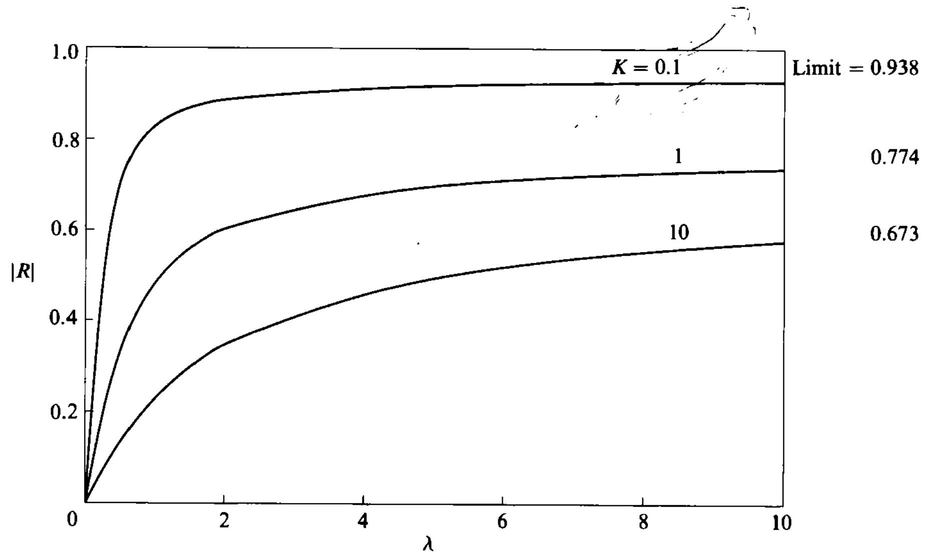

FIGURE 2. The amplitude of the surface elevation at the wavemaker for the same conditions as in figure 1.

of $\eta(0, t)$ are straightforward numerical tasks. Some numerical values of these two quantities as functions of λ for $h = 1$ and for three values of K are displayed in figures 1 and 2. The wave amplitude decreases monotonically as K increases. Each curve has a shallow minimum as a function of λ and approaches its limiting value as $\lambda \to \infty$ from below. It should be remembered that, when λ is finite and non-zero, there is some energy dissipation at the wavemaker which may account for the dip in the amplitude of the generated wave. The surface elevation at the wavemaker (figure 2) is a monotonic increasing function of λ and also decreases as K increases.

4. Infinite depth

When the fluid is of infinite depth, with a boundary at $x = 0$ of which the top portion, of depth h, is the wavemaker, the analysis can proceed in a similar fashion to that for a finite depth of fluid. A suitable form for ϕ_1 that satisfies the forcing condition (2.5) is given by

$$\phi_1 = \frac{2\sigma}{\pi(s+i\sigma)} \int_0^\infty \frac{1-\cos\kappa h}{\kappa^2} \sin\kappa z \exp(-\kappa x)\, d\kappa. \tag{4.1}$$

The pressure on the free surface from this part of the solution vanishes, and the vertical velocity $\hat{w}_1$ there is given by

$$\hat{w}_1 = \frac{2\sigma}{\pi(s+i\sigma)} \int_0^\infty \frac{1-\cos\kappa h}{\kappa} \exp(-\kappa x)\, d\kappa, \tag{4.2}$$

which can also be written as a Fourier integral in the form

$$\hat{w}_1 = \frac{2\sigma}{\pi(s+i\sigma)} \int_0^\infty \frac{1-e^{-kh}}{k} \cos kx\, dk. \tag{4.3}$$

The appropriate form for ϕ_2 is now

$$\phi_2 = \int_0^\infty \hat{A}(s,k) \cos kx\, e^{kz}\, dk. \tag{4.4}$$

Following the same steps as in the finite-depth case, we arrive at an equation for $\hat{B}$ to correspond to (3.11), namely

$$\left[\lambda + \frac{2Ks}{\pi} \int_0^\infty \frac{k}{s^2+\sigma_k^2}\, dk\right] \hat{B} = \frac{2\sigma s^2}{\pi(s+i\sigma)} \int_0^\infty \frac{1-e^{-kh}}{k(s^2+\sigma_k^2)}\, dk, \tag{4.5}$$

where now

$$\sigma_k^2 = k(1+Kk^2), \quad \sigma^2 = 1+K. \tag{4.6}$$

The transient motion near the contact line has the same form as in the finite-depth case, since the integrals in (4.5) for large s are similar to those in (3.11). For large t, we can determine the wave generated by the wavemaker in the same way as before and we find that the complex amplitude of the wave, denoted by R, is now given by

$$R = \frac{2\sigma^2(1-e^{-h})}{q'} \left[\frac{J_2' + \dfrac{i\lambda\pi}{2\sigma K}}{J_1' + \dfrac{i\pi}{q'} + \dfrac{i\lambda\pi}{2\sigma K}}\right], \tag{4.7}$$

where $q' = 1+3K$. This expression for R is very similar to that in the finite-depth case given by (3.19), but the integrals are now defined by

$$J_1' = \int_0^\infty \frac{k}{\sigma_k^2-\sigma^2}\, dk, \tag{4.8}$$

which is a principal-value integral, and by

$$J_2' = \int_0^\infty \frac{k^2-k'^2}{k(\sigma_k^2-\sigma^2)}\, dk, \tag{4.9}$$

where

$$k'^2 = \frac{1-e^{-kh}}{1-e^{-h}}. \tag{4.10}$$

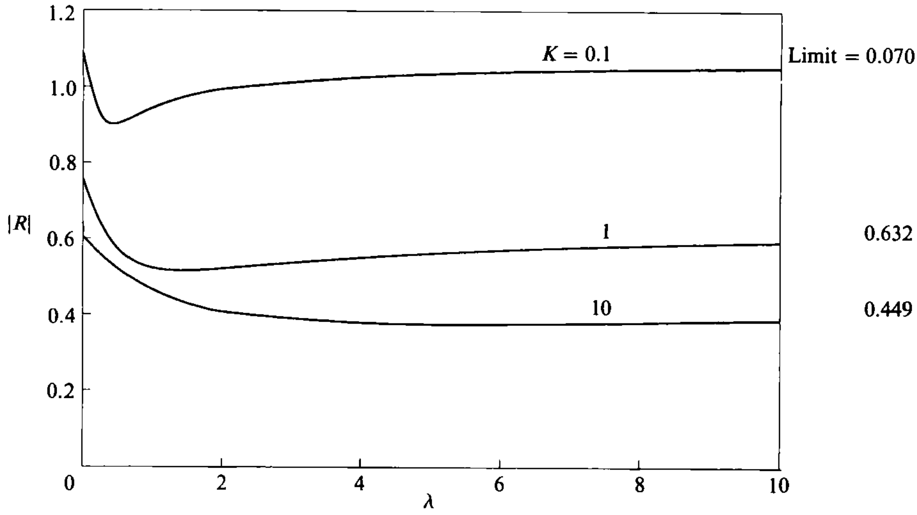

FIGURE 3. The amplitude of the wave generated by a wavemaker of depth $h = 1$ in fluid of infinite depth.

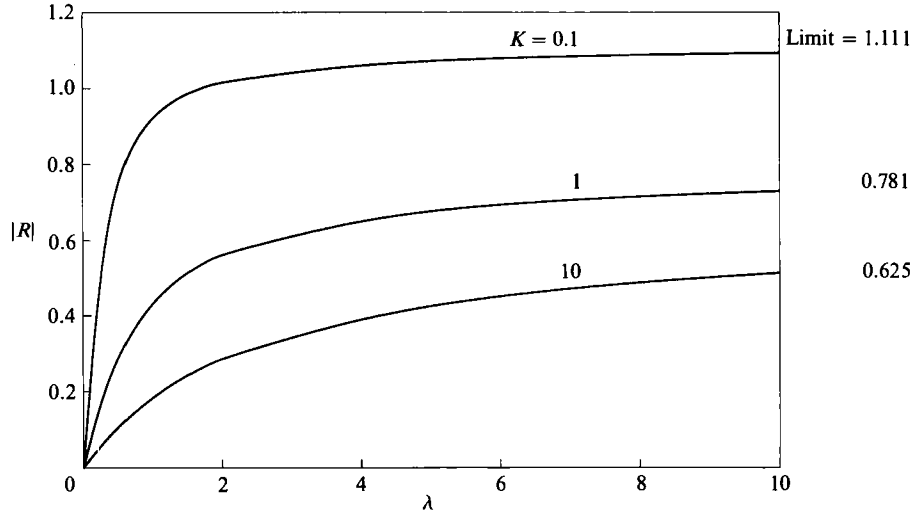

FIGURE 4. The amplitude of the surface elevation at the wavemaker for the same conditions as in figure 3.

The free surface elevation at the wavemaker is given by

$$\eta(0,t) = e^{-i\sigma t}\frac{\lambda\sigma}{K}(1-e^{-h})\left[\frac{J_1'-J_2'+\dfrac{i\pi}{q'}}{J_1'+\dfrac{i\pi}{q'}+\dfrac{i\lambda\pi}{2\sigma K}}\right]. \tag{4.11}$$

Numerical values of $|R|$ and of $\eta(0,t)$ as functions of λ for $h = 1$ and for various values of K are displayed in figures 3 and 4. The main features are similar to those of the corresponding results for the finite-depth case shown in figures 1 and 2. The minima of the wave amplitudes are somewhat more pronounced.

5. Conclusions

Two main results have been established in this paper. It has been shown that capillary–gravity waves generated by a wavemaker can be predicted from the known motion of the wavemaker, provided an appropriate edge condition is applied, and without assuming a prescribed slope of the free surface at the contact line. The special cases of a fixed contact angle and a contact line fixed on the wavemaker have been included. The solutions have been given for a particular motion of the wavemaker, namely an impulsively started harmonic oscillation, and the velocity of the wavemaker has been uniform over the immersed part of the wavemaker. Extensions to other time variations and to depth-dependent velocities can easily be made.

The second result has been to show that, with capillarity and an appropriate edge condition, the transient motion after an impulsive start does not introduce a singularity in either the position or the slope of the free surface. It is not necessary to ensure that the initial motion of the wavemaker is sufficiently smooth, nor do we need to include nonlinear effects to remove the singularity that occurs when surface tension is neglected.

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